

REYNOLDS' LIMIT FORMULA FOR DORODNITZYN'S  
ATMOSPHERIC BOUNDARY LAYER MODEL  
IN CONVECTIVE CONDITIONS

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**Abstract:** Atmospheric convection is an essential aspect of atmospheric movement, and it is a source of errors in Climate Models. Being able to generate approximate limit formulas and compare the estimations they produce, could give a way to reduce them. In this article, it is shown that it is enough to assume that the velocity's  $L^2$ -norm is bounded, has locally integrable,  $L^1_{loc}$ , weak partial derivatives up to order two, and a negligible variation of its first velocity's coordinate in direction parallel to the surface, to obtain a Reynolds' limit formula for a Dorodnitsyn's compressible gaseous Boundary Layer in atmospheric conditions.

**AMS Subject Classification:** 35Q30, 76N15, 76N20

**Key Words:** gas dynamics, boundary-layer theory, Navier-Stokes equations, Reynold's limit formulas

## 1. Introduction

A suitable approximate model for the air near the Earth's surface could tie

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both the free-stream velocity and the no slip condition. In Theorem 19, it will be shown that there is a Reynolds' limit formula:

$$f \frac{\partial^2 u}{\partial y^2} = \frac{\partial f}{\partial y} \frac{\partial u}{\partial y},$$

for a Dorodnitsyn's compressible Boundary Layer, where  $u$  is the first velocity's component,  $f = [1 - (u^2/2i_0)]^{-6/25}$ ,  $y$  denotes the height, and  $i_0$  is constant. In order to do so, we find an estimate, independent of the domain's scale:

$$\|\nabla F^\epsilon\|_{L^2(\Omega; \mathbb{R}^2)} \leq \frac{c_2 U^3}{2C}$$

for the  $L^2$ -norm of a corresponding incompressible vector field  $F^\epsilon$ , where  $U$  is the air's velocity over the Boundary Layer, and  $C$  is a constant set out by the rest of the boundary conditions given to the initial problem.

The solution procedure consists of three main steps. First, the application of Bayada and Chabat's change of variables transforms the original problem to an adimensional model where the effect of the small parameter of proportion,

$$\epsilon = \max \{h(x) \mid x \in [0, L]\} / L,$$

on each term, is explicit. Then, an adaptation of Dodordnitsyn's technique is applied to present it in an incompressible form, where Majda's Energy Method is used to obtain a bound that is independent of  $\epsilon$  for the  $L^2$ -norm of the incompressible gradient. Finally, we show that the family of solutions to the adimensional problem, indexed by the small parameter  $\epsilon$  is contained in a bounded set of a Sobolev space. Consequently, the Rellich-Kondrachov Compactness Theorem implies that the sequence of solutions has a subsequence that converges uniformly in the space  $L^2(\Omega)$  when the parameter  $\epsilon$  tends to zero.

### 1.1. Motivation

There is a need to lower biases in continental warmth to obtain better atmosphere models G.M. Martin et al. [14, p. 725]. The release of energy to the atmosphere by convective parcels contributes to these errors. Its calculation has historically been a way to reduce inaccuracies in surface temperature descriptions K. Stüwe [25, p. 59]. A temperature difference between a specific surface in contact with a gas and its surrounding neighborhood is the origin of a vertical draft of air, a natural convection air parcel. A sudden expansion of the gas in touch with the increased temperature gives a drop in its density, which

in turn makes it lighter B.R. Morton et al. [19] and A. Bouzinaoui et al. [4]. However, ascending air acceleration is modeled by compressible Navier-Stokes equations P.-L. Lions [15] and F. Boyer et al. [3]. The suggestion of this work is to overcome this difficulty by looking for Reynolds' limit formulas, deduced from compressible Boundary Layer models. In this article, a first Reynolds' limit formula is found for the Dorodnitsyn's ideal gas and constant total energy Boundary Layer model, which admits an incompressible adimensional presentation where the evolution parameter problem is stated and the convective non linear term estimated through its *free-stream* velocity value.

## 1.2. Statement of the Problem

An atmospheric gas is a *newtonian fluid*, which implies the use of compressible Navier-Stokes equations P.-L. Lions [15]. If, instead of considering a Boundary Layer, a two-dimensional incompressible Navier-Stokes model is applied to study the behaviour of a liquid in contact with a solid surface, then there exist a smooth solution for each given viscosity value. For a fixed initial condition, a set of viscosity values has a corresponding family of well defined classical solutions. When the viscosity tends to zero, this family of solutions converges to an Euler's Equations solution with the same initial condition A.J. Majda et al. [13].

However, even in the simplest case of an incompressible flow whose vorticity is zero everywhere on its domain, an Euler's solution satisfying the condition of null velocity at  $\Gamma_0$ , has a null velocity throughout the whole domain C.V. Valencia [27, p. 19]. Therefore, there are no two-dimensional Euler solutions with zero vorticity that comply with both the positive horizontal component of velocity at the top of the domain and the no slip condition at its bottom H. Schlichting et al. [23, p. 145]. This motivates the statement of a Boundary Layer model to more appropriately depict this phenomenon. Moreover, numerical approximations of boundary layer solutions describe velocity profiles similar to those found in reality H. Schlichting [22, p. 143].

In 1935, A. Busemann [6] proposed the first compressible Boundary Layer model to represent the behaviour of a gas with upper outflow velocity smaller than the velocity of sound, and Prandtl number equal to one. In his model, pressure terms are discarded, but temperature, viscosity, and density vary in accordance with ideal gas empirical properties to more accurately describe an atmospheric boundary layer moving over a surface. He presents temperature as a function of velocity, and employs it to describe the rest of the state variables in terms of velocity as well.

Busemann's model considered a power-law between viscosity and temperature whose exponent was later corrected in T. von Kármán and H.S. Tsien [26] article of 1938, where they developed a different method of solution for the same problem. Less than a decade later, in 1942, A.A. Dorodnitsyn [9] postulated a similar model, but allowed pressure to vary with  $x$ , which could imply the Boundary Layer to be separated from the surface. In this work, he defined several changes of variables. The first one of these allowed him to write the compressible model as an incompressible system. Here, we adapt this coordinates' change to a similar but not rectangular adimensional domain that will be obtained from  $\Omega_h$ , and defined in Theorem 15.

Limit formulas for a small parameter of proportion have their origin in O. Reynolds' [20] article "*On the Theory of Lubrication and Its Application to Mr. Beauchamp Tower's Experiments, Including an Experimental Determination of the Viscosity of Olive Oil*", published in 1886. Reynolds' Formula was extensively used without a formal proof that it was indeed Navier-Stokes Equations' limit when the small parameter of proportion between the domain's height and its length tends to zero. This was accomplished a hundred years later by G. Bayada and M. Chambat [1] for Stokes' Equations.

In 2009, L. Chupin and R. Sart [8] successfully showed, through an application of Didier Bresch and Benoît Desjardin's Entropy Methods, that the compressible Reynolds equation is an approximation of compressible Navier-Stokes equations. For a thin domain filled with gas, the authors mention that there appears to be only one result of this type of problem. This is due to E. Marusic-Paloka and M. Starcevic [17], [18]. Marusic-Paloka and Starcevic show the convergence of a two-dimensional compressible Stokes Equations.

In the literature, it does not seem to exist a small parameter asymptotic analysis for a compressible gaseous Boundary Layer model with a convective non linear term, such as Dorodnitsyn's Model, nor an adaptation of Dorodnitsyn's change of variables to this particular domain's shape to find a limit formula for a compressible case in terms of an incompressible expression. The main result of this study is stated in Theorem 18 and proved in Subsection 2.3. Meanwhile, it can be expressed by the following assertion: Dorodnitsyn's Model may be approximated by a limit formula.

### 1.3. The Domain

Laminarity – and therefore two-dimensionality of the domain – in the liquid's movement when it is in contact with a solid surface is a supposition based on experimental observations T. von Kármán et al. [26] and S. Goldstein [12],

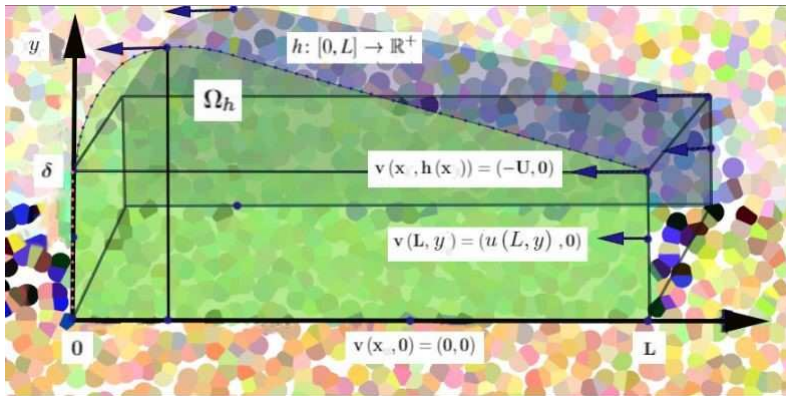


Figure 1: The Domain  $\Omega_h$

and it is still regarded as a good assumption to describe it at an initial stage of a Boundary Layers' motion K. Gersten [11, p. 11] and S. Goldstein [12]. Here, the Boundary Layer is represented as a two-dimensional slice where the convective bubble is beginning to form although it has not yet separated from the surface, and it is slightly different from the rectangle that constitutes the domain in Dorodnitsyn's model.

**Definition 1.** Let  $h: [0, L] \rightarrow (0, \infty)$  be a smooth function such that  $h(0) = h(L) = \delta$ . The curve  $h$  is assumed to be twice differentiable in the interval  $(0, L)$  with well defined continuous extensions for itself and its derivatives to  $\{0\}$  and  $\{L\}$ , i.e.  $h \in C^2([0, L]; (0, \infty))$ , and to have only one critical point which is a maximum. Moreover, suppose  $L > 0$ . Then, the domain is denoted as:

$$\Omega_h := \{(x, y) \in \mathbb{R}^2 \mid 0 < x < L \ \& \ 0 < y < h(x)\}.$$

The domain's topological boundary,  $\partial\Omega_h$ , is drawn by the union of the segments:  $\Gamma_0 = \{(x, 0) \in \mathbb{R}^2 ; 0 \leq x \leq L\}$ ,  $\Lambda_0 = \{(0, y) \in \mathbb{R}^2 ; 0 \leq y \leq \delta\}$ ,  $\Lambda_L = \{(L, y) \in \mathbb{R}^2 ; 0 \leq y \leq \delta\}$ , and the curve

$$\Gamma_h = \{(x, h(x)) \in \mathbb{R}^2 ; 0 \leq x \leq L\}.$$

**Remark 2.** The vector  $-\mathbf{e}_1 = (-1, 0) \in \mathbb{R}^2$  depicts the wind's direction above the Boundary Layer  $\Omega_h$ . Likewise,  $\mathbf{e}_3 = (0, 1) \in \mathbb{R}^2$  portrays the direction from the Earth's surface to its atmosphere. Similarly, the length  $L > 0$  is a

fixed real number which represents the distance covered by the *free-stream* in direction  $-\mathbf{e}_1 = (-1, 0)$  over  $\Gamma_h$ . On the other hand, continuation of trajectories in the Boundary Layer is broken if for some  $x \in [0, L]$  there is a pressure drop that generates a lift, a separation of the volume of the air from the surface. At that moment, the phenomenon's description in terms of a fixed domain is no longer possible.

#### 1.4. Dorodnitsyn's Model Equations

**Definition 3.** Let  $\Omega_h$  be as in Definition 1,  $\rho \in L^1(\Omega_h; (0, \infty))$  be the *density*; the *velocity*,  $\mathbf{v} = (u, v) \in L^2(\Omega_h; \mathbb{R}^2) \cap L^1_{loc}(\Omega_h; \mathbb{R}^2)$ ; the *absolute temperature*,  $T \in L^1_{loc}(\Omega_h; (0, \infty))$ ; the *pressure*,  $p \in L^1_{loc}(\Omega_h)$ ; the *dynamic viscosity*,  $\mu \in L^1_{loc}(\Omega_h)$ ; and the *thermal conductivity* be  $\kappa \in L^1_{loc}(\Omega_h)$ ; all with well defined first order weak partial derivatives, locally integrable in the Lebesgue sense, *i.e.* in  $L^1_{loc}(\Omega_h)$ .

Dorodnitsyn's model is formed by seven equations given for the seven variables  $\rho, u, v, T, p, \kappa, \mu$ , described above. The first three come from the conservation laws of Newtonian fluids: the stationary Conservation of Mass Law, F. Boyer et al. [3], Eq. (1), the compressible Boundary Layer Conservation of Momentum Law, A. Dorodnitsyn [9], Eq. (2), and the simplified Conservation of Energy per Unit Mass Law, Eq. (9), that is obtained in Proposition 5 from an application of L. Crocco's [7] procedure to the stationary and approximated, Conservation of Energy Law stated below as Eq. (3).

Consider:

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0; \quad (1)$$

$$\rho \left( u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial y} \left( \mu \frac{\partial u}{\partial y} \right); \text{ and} \quad (2)$$

$$\rho \left[ u \frac{\partial(c_p T)}{\partial x} + v \frac{\partial(c_p T)}{\partial y} \right] = \frac{\partial}{\partial y} \left[ \kappa \frac{\partial T}{\partial y} \right] + \mu \left( \frac{\partial u}{\partial y} \right)^2 + \frac{\partial p}{\partial t}, \quad (3)$$

where  $c_p$  is the *specific heat transfer coefficient at constant pressure*.

The next four are Ideal Gases properties and empirical laws. In general, the dynamic viscosity  $\mu$  satisfies the proportionality relation

$$Pr = \frac{c_p \mu}{\kappa}$$

for a *thermal conductivity*  $\kappa$  and a Prandtl number  $Pr$ . In this case, assume  $Pr = 1$ . This is:

$$1 = \frac{c_p \mu}{\kappa}; \tag{4}$$

and, the *Equation of State* K. Saha [21, p. 24],

$$pV = n R^* T; \tag{5}$$

where  $R^*$  is the *Universal Gas Constant*,  $n$  is the number of moles in a volume  $V$ , and  $V = V(B_r) = \iiint_{B_r} d\mathbf{x}$  where

$$B_r = \{\mathbf{x} = (x, y, z) \in \mathbb{R}^3 ; \|\mathbf{x} - \hat{\mathbf{x}}\| < r\} \subset \mathbb{R}^3,$$

for a given point  $\hat{\mathbf{x}} \in \Omega_h \cap \mathbb{R}^3$  and a value  $r > 0$  such that  $\Omega_h \subset B_r$ . This last equation is also used by Dorodnitzyn in the form:

$$\rho = \frac{p}{RT}, \tag{6}$$

for  $R = R^*/M$ , where  $M$  is the *molecular weight* of the gas.

The *adiabatic polytropic atmosphere*, O.G. Tietjens [28, p. 35], is a relation:

$$pV^b = c; \tag{7}$$

where  $b \cong 1.405$ ,  $c$  are fixed constants, and  $V$  has the value described above.

Finally, given two values  $\mu_0$  and  $T_0$  of  $\mu$  and  $T$  at the same point  $(x_0, y_0) \in \Omega_h$ , there is a *Power-Law*, A.J. Smits et al. [24, p. 46]:

$$\frac{\mu}{\mu_0} = \left(\frac{T}{T_0}\right)^{\frac{19}{25}}. \tag{8}$$

**Remark 4.** First of all, when the air flow moves over a plane surface, has a velocity lower than the velocity of sound, and the surface has a homogeneous temperature, the Prandtl number is equal to 1, H. Schlichting et al. [23, p. 215], Eq. (4), and  $c_p \mu$  replaces  $\kappa$  in Eq. (3). Second, a gas in the range of temperatures and densities found in the Earth's atmosphere fulfills the premises discovered for an Ideal Gas, P.-L. Lions [15, p. 8], such as the Equation of State, Eq. (5). Moreover, when air moves in a convective parcel, the process is fast enough to expect that there will not be a heat transfer between the gas within the convective draft and its environment. Thus, adiabatic conditions imply another association, known as an adiabatic polytropic atmosphere, O.G. Tietjens [28, p. 35]. Additionally, in a temperature range of [150, 500] Kelvin, there is a Power-Law between dynamic viscosity and  $T$ , A.J. Smits et al. [24, p. 46].

One can follow L. Crocco's [7] procedure to find a Conservation of Energy Law from which Dorodnitzyn's model equation Eq. (9) is deduced, and find that it is equivalent to Eq. (3) when the Prandtl number is equal to 1, Eq. (4), as it is outlined in the following paragraph.

**Proposition 5.** *Let  $\rho, u, v, T, p, \kappa, \mu$  be as they were described in Definition 3. Then, they satisfy Eq. (3) if and only:*

$$\rho \left[ u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} \right] \left( c_p T + \frac{u^2}{2} \right) = \frac{\partial}{\partial y} \left[ \mu \frac{\partial}{\partial y} \left( c_p T + \frac{u^2}{2} \right) \right]. \quad (9)$$

*Proof.* First, Eq. (4) allows to make the substitution  $\kappa = c_p \mu$  in the right side of Eq. (3). This way one can arrive at:

$$\rho \left[ u \frac{\partial (c_p T)}{\partial x} + v \frac{\partial (c_p T)}{\partial y} \right] = \frac{\partial}{\partial y} \left[ \mu \frac{\partial (c_p T)}{\partial y} \right] + \mu \left( \frac{\partial u}{\partial y} \right)^2 + \frac{\partial p}{\partial t}. \quad (10)$$

Also, the product of the first velocity coordinate  $u$  and Eq. (2) gives:

$$\rho \left[ u \frac{\partial}{\partial x} \left( \frac{u^2}{2} \right) + v \frac{\partial}{\partial y} \left( \frac{u^2}{2} \right) \right] = u \frac{\partial}{\partial y} \left( \mu \frac{\partial u}{\partial y} \right) - u \frac{\partial p}{\partial x}. \quad (11)$$

Finally, Eq. (9) is obtained from the addition of Eq. (10) and (11) because  $\partial p / \partial t = (\partial p / \partial x) (\partial x / \partial t) = u (\partial p / \partial x)$ .  $\square$

**Remark 6.** It is possible to notice in Eq. (9) that in Dorodnitzyn's model, the *kinetic energy* generated by the velocity coordinate  $v$  in the orthogonal direction to the surface is taken as negligible; the *total energy per unit mass*,  $E = c_p T + u^2/2$ , is considered the addition of the *kinetic energy per unit mass*  $u^2/2$  and the *internal energy* in terms of *specific enthalpy*  $e = c_p T$ .

### 1.5. Dorodnitzyn's Model Boundary Conditions

The velocity at the upper top  $\Gamma_h$  of  $\partial\Omega_h$  is called the *free-stream velocity*. Let:

$$\mathbf{v}|_{\Gamma_h} = (-U, 0), \quad (12)$$

for a strictly positive constant real value  $U > 0$ . Also, the velocity value at the lower lid  $\Gamma_0$  is:

$$\mathbf{v}|_{\Gamma_0} = (0, 0). \quad (13)$$



Similarly, a constant *free-stream temperature*,

$$T|_{\Gamma_h} = T_h > 0, \quad (14)$$

and a homogeneous *free-stream dynamic viscosity* value

$$\mu|_{\Gamma_h} = \mu_h > 0, \quad (15)$$

are given in  $\Gamma_h$ .

Furthermore, there are periodic velocity conditions at the vertical segments of the boundary,  $\Lambda_0$  and  $\Lambda_L$ , described in the Definition 1. This is: For all  $y \in (0, \delta)$ ,

$$(u(0, y), 0) = (u(L, y), 0). \quad (16)$$

Finally, we have a Neumann condition for  $T$ : For all  $x \in [0, L]$ ,

$$\frac{\partial T}{\partial y}(x, 0) = 0. \quad (17)$$

**Remark 7.** This last condition represents an adiabatic wall in the surface  $\Gamma_0$ . If the wind's velocity is less than the velocity of sound, the gas adheres to the solid surface, T. von Kármán et al. [26] and A.J. Smits et al. [24, p. 52]. This is called the *no slip* condition, as seen in Eq. (13). On the other hand, there is a logarithmic wind velocity profile on the Earth's troposphere that depends on the type of atmosphere, and is not valid close to the Earth's surface, but provides a boundary condition  $U$  at the upper top  $\Gamma_h$  of  $\Omega_h$ . For example, the classical Fleagle and Businger's [10, p. 274] Atmospheric Physics book reports a horizontal velocity measurement of 4 m/s at a height of 0.4 m,  $\mathbf{v}(0, 0.4) = (4, 0)$ , in an *unstable* atmosphere at O'Neill, Nebraska on 19 August 1953. Moreover, this value and the free-stream temperature determine that of the surface temperature, as will be shown in the following Lemma 8, Eq. (19). Similarly, the pressure  $p|_{\Gamma_0} = p_0$  can be known from  $U$  and  $T_h$  through Eq. (5) and (19). Once the density is expressed in terms of the velocity  $u$ , as in Lemma 11,  $\rho|_{\Gamma_0} = \rho_0$  can be calculated. Finally, Eq. (4) provides a way to obtain  $\mu_h$  from a surface value of  $\kappa_h$  given by the material.

## 2. Limit Formula

### 2.1. Adimensional Model

**Lemma 8.** *Let  $\rho, u, v, T, p, \kappa, \mu$  be as in Definition 3. If the no slip condition (13) is satisfied, then Eq. (9) has a constant solution  $E = c_p T_h + U^2/2$  in the domain  $\Omega_h$ , described in Definition 1, that fulfills the remaining boundary conditions (12), (14), and (17) given for  $u, v$ , and  $T$ .*

*Proof.* It is enough to substitute the constant value  $E = c_p T_h + U^2/2$  in Eq. (9) to see that both sides become zero. Because  $E = c_p T + u^2/2$ , this allows us to express the absolute temperature in the form

$$T(u) = T_h + \frac{1}{2c_p} (U^2 - u^2). \quad (18)$$

The boundary conditions (12) and (14) are verified by construction. If  $y = 0$ , Eq. (18) and the no slip condition (13) imply that:

$$T|_{\Gamma_0} = T_h + \left(1 - \frac{U^2}{2c_p}\right). \quad (19)$$

Thus,  $T|_{\Gamma_0} = T_0 > 0$ , and the boundary condition (17) is fulfilled.  $\square$

**Corollary 9.** *Under the same assumptions, where the free-stream temperature  $T_h > 0$ , as is stated in (14),  $T$  can be seen in terms of  $T_0$  as:*

$$T(u) = T_0 \left(1 - \frac{u^2}{2c_p T_0}\right). \quad (20)$$

*Proof.* The previous Lemma 8 shows that the total energy  $E$  has a constant value throughout the domain. We can use the no slip condition (13) in the expression  $E = c_p T + u^2/2$  to obtain a new way to calculate it as  $E = c_p T_0$ . Hence,  $c_p T_0 = c_p T + u^2/2$ , and we get Eq. (20).  $\square$

**Remark 10.** By Definition 3, the absolute temperature  $T > 0$  in the domain  $\Omega_h$  as described in Definition 1. Additionally  $c_p > 0$ . In consequence, the total energy per mass unit  $i_0 := c_p T_0 = c_p T + u^2/2$  is strictly bigger than the kinetic energy  $u^2/2$  generated by the first velocity's component. Therefore the difference  $1 - (u^2/2i_0) \neq 0$  in  $\Omega_h$ .

**Lemma 11.** *Once again, let  $\rho, u, v, T, p, \kappa, \mu$  be as in Definition 3. Suppose that Eq. (3), (4), (5), (7) and (8) are satisfied by  $\rho, u, v, T, p, \kappa,$  and  $\mu$  in  $\Omega_h$  with the boundary conditions (12), (13), (14), and (15). Then:*

$$p(u) = c_1 [1 - (u^2/2i_0)]^{\frac{b}{(b-1)}}; \tag{21}$$

$$\rho(u) = c_2 \frac{[1 - (u^2/2i_0)]^{\frac{b}{(b-1)}}}{[1 - (u^2/2i_0)]}; \text{ and} \tag{22}$$

$$\mu(u) = c_3 [1 - (u^2/2i_0)]^{\frac{19}{25}}; \tag{23}$$

where  $c_1 = p_0 T_0^{\frac{2b}{(b-1)}}$ ,  $c_2 = c_1 R^{-1} T_0^{-1}$ , and  $c_3 = \mu_h T_h^{-\frac{19}{25}} T_0^{\frac{19}{25}}$ .

*Proof.* As previously seen in Proposition 5, Eq. (3) and (4) are equivalent to Eq. (9). If the boundary conditions (12) and (13) are known, the Lemma 8 gives  $T_0 > 0$  at  $\Gamma_0$ , Eq. (19), and the expression of temperature in terms of  $u$ , Eq. (20). Then, Eq. (5) provides a value  $p|_{\Gamma_0} = p_0 = (n R^* T_0)/V > 0$ . Analogously, regarding Eq. (5), (20), and the last Remark 10, we have that  $p \neq 0$  in  $\Omega_h$ . Thus, from Eq. (7), we get  $p_0 [(n R^* T_0)/p_0]^b = p [(n R^* T)/p]^b$ . This is,  $p = p_0 T_0^{2b/(b-1)} T^{b/(b-1)}$ . The substitution of Eq. (20) in this last expression conduces to (21). Similarly, Eq. (21), Eq. (20), and Eq. (6), which is equivalent to Eq. (5), conduces to (22). Finally, Eq. (23) is a consequence of Eq. (8), Eq. (20), and the value  $\mu_h$  of (15). □

**Remark 12.** Atmospheric pressure is regarded as the weight impressed by the column of air over a point  $x$  at its base, O.G. Tietjens [28, p. 18]. Dorodnitzyn assumes  $p$  to be dependent only of  $x$ , and that for each  $x \in (0, L)$ ,  $p(x, y)$  is given by its corresponding value  $p(x, h(x))$  at  $\Gamma_h$ . In the Corollary 13, we emphasize that this can be seen as a consequence of temperature's observed linear decrease with height from the Earth's surface to the troposphere's upper border, K. Saha [21, p. 20]. Moreover, this allows us to consider a constant pressure value determined by the free-stream velocity in Theorem 15 below.

**Corollary 13.** *Under the same conditions as in Lemma 11, let  $p(x, y) = g \int_y^\infty \rho(x, z) dz$  for all  $(x, y) \in \Omega_h$ , where  $g$  is the standard gravity constant. If, additionally,  $\beta > 0$  is such that  $T(x, y) = T_0 - \beta y \forall (x, y) \in \Omega_h$ , then for all  $(x, y) \in \Omega_h$ :*

$$p(x, y) \cong c_1 [1 - (U^2/2i_0)]^{\frac{b}{(b-1)}}; \text{ and} \tag{24}$$

$$\rho(x, y) \cong c_2 \frac{[1 - (U^2/2i_0)]^{\frac{b}{(b-1)}}}{[1 - (u^2(x, y)/2i_0)]}. \tag{25}$$

*Proof.* From Lemma 11, we have  $T_0 > 0$ , Eq. (21) and (22). If  $T(x, y) = T_0 - \beta y$  is substituted in Eq. (16), that is equivalent to the given Eq. (15), and the corresponding density expression is used in the atmospheric pressure’s definition  $p(x, y) = g \int_y^\infty \rho(x, z) dz$ . Then,  $\ln(p(x, y)) - \ln(p_0) = g\beta [\ln((T_0 - \beta y)/T_0)]$ . For this reason, if  $y$  is sufficiently small for the term  $\beta y$  to be discarded, the variation of pressure with height may be negligible. Hence,  $p$  can be approximated by its value in each  $(x, h(x)) \in \Gamma_h$ . The Eq. (21) with values in  $\Gamma_h$  implies Eq. (24). Furthermore, Eq. (25) is inferred from Eq. (24) and (22).  $\square$

**Lemma 14.** *Let  $h$  and  $\Omega_h$  be as in Definition 3, and  $\rho, u, v, T, p, \kappa, \mu$  as in Definition 3. For each  $L > 0$  and  $H := \max\{h(x) \mid x \in [0, L]\}$ , there are a parameter  $\epsilon := H/L > 0$ , and a diffeomorphism  $\phi^\epsilon: \Omega_h \rightarrow \Omega_\epsilon$ ,  $\phi^\epsilon(x, y) = (s, \tau) := (x/L, y/(L\epsilon))$  for all  $(x, y) \in \Omega_h$ . Also, there is a vector field  $\mathbf{v}^\epsilon = (u^\epsilon, v^\epsilon) \in L^2(\Omega_\epsilon; \mathbb{R}^2) \cap L^1_{loc}(\Omega_\epsilon; \mathbb{R}^2)$  such that  $\mathbf{v}^\epsilon(s, \tau) = (u^\epsilon(s, \tau), v^\epsilon(s, \tau))$  with  $u^\epsilon(s, \tau) = (1/L)u(Ls, L\epsilon\tau)$ ,  $v^\epsilon(s, \tau) = (1/(L\epsilon))v(Ls, L\epsilon\tau)$ ; a density  $\rho^\epsilon \in L^1(\Omega_\epsilon; (0, \infty))$ ,  $\rho^\epsilon(s, \tau) := c_2[\sigma_0]^{b/(b-1)}\sigma^{-1}(s, \tau)$ , where  $\sigma$  denotes  $\sigma(s, \tau) = 1 - ([Lu^\epsilon(s, \tau)]^2/2i_0)$ ,  $\sigma_0$  is the number  $1 - ([LU^\epsilon]^2/2i_0)$ , and  $U^\epsilon = (1/L)U$  is the free-stream velocity on the curve  $h^\epsilon \in C^2([0, 1])$  such that  $h^\epsilon(x) := h(Ls)/(L\epsilon)$ . Analogously, there is a dynamic viscosity  $\mu^\epsilon \in L^1_{loc}(\Omega_\epsilon)$  with  $\mu^\epsilon := c_3\sigma^{\frac{19}{25}}$ .*

*Proof.* First of all,  $\phi^\epsilon$  is linear. Because  $\text{Ker}(\phi) = \{(0, 0)\}$ , it is invertible. Its Jacobian determinant is  $|D\phi^\epsilon| = 1/(L^2\epsilon) > 0$ . Consequently, by the Inverse Function Theorem,  $\phi^\epsilon$  is a diffeomorphism of  $\Omega_h$ . Second, the vector field is obtained via the Chain Rule: Let  $t \in [0, \infty)$  be the time, then  $u^\epsilon = \partial s/\partial t = (\partial s/\partial x)(\partial x/\partial t) = (1/L)u$ . Similarly, we obtain  $v^\epsilon$  and the free-stream velocity  $U^\epsilon$ . Moreover, if  $u \in L^2(\Omega_h)$ ,

$$\|u\|_{L^2(\Omega_h)}^2 = \iint_{\Omega_h} u^2(x, y) dx dy = L^2\epsilon \iint_{\Omega_\epsilon} [Lu^\epsilon]^2(s, \tau) ds d\tau.$$

So that,

$$\|u^\epsilon\|_{L^2(\Omega_\epsilon)}^2 = L^4\epsilon\|u\|_{L^2(\Omega_h)}^2 < \infty, \tag{26}$$

and  $u^\epsilon \in L^2(\Omega_\epsilon)$ . In the same way,  $u^\epsilon \in L^1_{loc}(\Omega_\epsilon)$ , and  $v^\epsilon \in L^1_{loc}(\Omega_\epsilon) \cap L^2(\Omega_\epsilon)$ . Finally, the density  $\rho^\epsilon$ , the curve  $h^\epsilon$ , and the dynamic viscosity  $\mu^\epsilon$  are determined by the corresponding commutative diagrams with  $\phi^\epsilon$ .  $\square$

**Theorem 15 (Adimensional Model).** *Let  $\rho, u, v, T, p, \kappa, \mu$  be as in Definition 3. Suppose they satisfy the Dorodnitzyn's Boundary Layer Model given by equations (1), (2), (3), (4), (5), (7), (8) with boundary conditions (12), (13), (14), (15), (16), (17). Additionally, assume  $p = c_1 [1 - (U^2/2i_0)]^{\frac{b}{(b-1)}}$  in  $\Omega_h$ . Then,  $u^\epsilon$ , and  $v^\epsilon$ , as defined in the Lemma 14 above, verify the following system in  $\Omega_\epsilon$ :*

$$\operatorname{div}(\rho^\epsilon u^\epsilon, \rho^\epsilon v^\epsilon) = 0; \text{ and} \tag{27}$$

$$L^2 \epsilon^2 \rho^\epsilon \left( u^\epsilon \frac{\partial u^\epsilon}{\partial s} + v^\epsilon \frac{\partial u^\epsilon}{\partial \tau} \right) = c_3 \frac{\partial}{\partial \tau} \left[ \sigma^{\frac{19}{25}} \frac{\partial u^\epsilon}{\partial \tau} \right], \tag{28}$$

with boundary conditions:

$$(u^\epsilon, v^\epsilon)|_{\phi^\epsilon(\Gamma_0)} = (0, 0); \tag{29}$$

$$(u^\epsilon, v^\epsilon)|_{\phi^\epsilon(\Gamma_h)} = (-LU^\epsilon, 0); \text{ and} \tag{30}$$

$$(u^\epsilon(0, \tau), 0) = (u^\epsilon(1, \tau), 0), \forall \tau \in [0, \delta/(L\epsilon)]; \tag{31}$$

where  $\rho^\epsilon$  and  $\sigma$  depend of  $u^\epsilon$ , in the way described in Lemma 14.

*Proof.* Considering the new directions, the generalized partial derivatives  $\partial u/\partial x = \partial u^\epsilon/\partial s$ ;  $\partial u/\partial y = (1/\epsilon) \partial u^\epsilon/\partial \tau$ ;  $\partial v/\partial y = \partial v^\epsilon/\partial \tau$ ;  $u(\partial u/\partial x) = L u^\epsilon (\partial u^\epsilon/\partial s)$ ; and  $v(\partial u/\partial y) = L v^\epsilon (\partial u^\epsilon/\partial \tau)$ . The weak derivative  $\partial p/\partial x = 0$  because  $U$  is constant. Similarly,  $p$  allows us to see

$$\rho = c_2 [1 - (U^2/2i_0)]^{\frac{b}{(b-1)}} / [1 - (u^2(x, y)/2i_0)].$$

Therefore,  $\partial/\partial y [\mu(\partial u/\partial y)] = L^{-1} \epsilon^{-2} c_3 \partial/\partial \tau \left[ \sigma^{\frac{19}{25}} (\partial u^\epsilon/\partial \tau) \right]$ . Finally, each term is substituted on each side of Eq. (1) and (2) to obtain Eq. (27) and (28). □

### 2.2. Incompressible Model

The domain's shape  $\Omega_h$  described in Definition 1 is different from the rectangular one in the original Dorodnitzyn's article. In addition, there is no domain  $\Omega_\epsilon$  in Dorodnitzyn's work, because this was obtained with the application of Bayada and Chambat's diffeomorphism  $\phi^\epsilon$ . Therefore, it is necessary to make an adjustment on Dorodnitzyn's change of variables to take into account the points  $(s, \tau) \in \Omega_\epsilon$  over a height  $\phi^\epsilon(0, \delta) = (0, \delta/(L\epsilon))$ , as is done in Eq. (33) below. This new diffeomorphism allows us to take the Adimensional Model into an incompressible form.

**Lemma 16.** *Let  $h \in C^2([0, L], (0, \infty))$  have only one critical point which is a maximum. Let  $\Omega_\epsilon$  and  $\rho^\epsilon$  be as described in Lemma 14. Suppose that the weak derivative  $\partial u/\partial x = 0$  a.e. in  $\Omega_h$ . Then, there is a diffeomorphism  $\eta = (\eta_1, \eta_2): \Omega_\epsilon \rightarrow \mathbb{R}^2$  such that  $\forall s \in [0, 1]$ :*

$$\eta_1(s, \tau) = \int_0^s \frac{1}{\rho^\epsilon(\zeta, \tau)} d\zeta, \quad \forall \tau \in [0, \delta/(L\epsilon)]; \tag{32}$$

$$\eta_1(s, \tau) = \int_{\tilde{s}}^s \frac{1}{\rho^\epsilon(\zeta, \tau)} d\zeta, \quad \forall \tau \in h^\epsilon([0, 1]); \text{ and} \tag{33}$$

$$\eta_2(s, \tau) = \int_0^\tau \rho^\epsilon(s, \xi) d\xi; \tag{34}$$

where  $\tilde{s}$  is the preimage of  $\tau = h^\epsilon(\tilde{s}) \in h^\epsilon([0, 1])$  such that the slope  $\partial h^\epsilon/\partial s(\tilde{s}) \geq 0$ .

*Proof.* By definition,  $\forall (s, \tau) \in \Omega_\epsilon$ ,  $\rho^\epsilon(s, \tau) = \rho(Ls, L\epsilon\tau) > 0$ . From the Remark 10, we know that  $\sigma$  is positive and bounded by 1. In addition, if  $h$  has one unique critical maximum in its domain,  $h^\epsilon$  does as well. In fact, the top cover of  $\Omega_\epsilon$  is given by the curve  $h^\epsilon$ , where each image  $\tau = h^\epsilon(s)$ , different from its cusp, has exactly two preimages, one of them on the ascending part of the curve where  $\partial h^\epsilon/\partial s(\tilde{s}) \geq 0$ . So that the horizontal segment  $(\tilde{s}, s) \times \{\tau\}$  is contained in  $\Omega_\epsilon$ . Thus, each Riemann integral  $\eta_1(s, \tau)$  is the limit of an of increasing and bounded sequence of Darboux sums which add positive values taken by the function  $\sigma$  over a horizontal and bounded segment contained in  $\Omega_\epsilon$ . As a result, for each  $(s, \tau) \in \Omega_\epsilon$ , the sequence of sums converges and  $\eta_1$  is well defined. In addition, Remark 10 implies that  $\sigma$  is strictly positive. Then,  $\eta_2$  is a well defined function in  $\Omega_\epsilon$ . Two of its partial derivatives are  $\partial\eta_1/\partial s = 1/\rho^\epsilon$ , and  $\partial\eta_2/\partial\tau = \rho^\epsilon$ . By the Monotone Convergence Theorem, if  $(\partial u/\partial x) = 0$  a. e. in  $\Omega_h$ , we calculate the product  $(\partial\eta_1/\partial\tau)(\partial\eta_2/\partial s) = 0$ . Then, the Jacobian determinant  $|D\eta| = 1$ . Hence, by the Inverse Function Theorem,  $\eta$  is a diffeomorphism of  $\Omega_\epsilon$ . □

**Theorem 17 (Incompressible Model).** *Let  $\rho, u, v, T, p, \kappa, \mu$  be as in Definition 3. Suppose they satisfy the Dorodnitsyn’s Boundary Layer Model given by equations (1), (2), (3), (4), (5), (7), (8) with boundary conditions (12), (13), (14), (15), (16), (17),  $p(x, y) = p(x, h(x)) \forall (x, y) \in \Omega_h$ , and  $\partial u/\partial x = 0$  a.e. in  $\Omega_h$ . Consider  $u^\epsilon, v^\epsilon, \rho^\epsilon, \sigma$ , and  $\sigma_0$  as in Lemma 14, and the domain  $\eta(\Omega_\epsilon) = \Omega$  as defined in Lemma 16. Then, there exists a stream-function  $\psi$  such that  $\partial\psi/\partial s = -\rho^\epsilon v^\epsilon$ ,  $\partial\psi/\partial\tau = \rho^\epsilon u^\epsilon$ ; and a vector field  $F^\epsilon = (F_1^\epsilon, F_2^\epsilon) \in$*

$L^2(\Omega; \mathbb{R}^2) \cap L^1_{loc}(\Omega; \mathbb{R}^2)$ ,  $F_1^\epsilon = \partial\psi/\partial\eta_2$  and  $F_2^\epsilon = -\partial\psi/\partial\eta_1$ , that satisfies:

$$\operatorname{div}(F_1^\epsilon, F_2^\epsilon) = 0 \quad \text{and} \tag{35}$$

$$L^2\epsilon^2 \left\{ F_1^\epsilon \frac{\partial F_1^\epsilon}{\partial \eta_1} + F_2^\epsilon \frac{\partial F_1^\epsilon}{\partial \eta_2} \right\} = C\tilde{\sigma}^{-1} \frac{\partial}{\partial \eta_2} \left[ \tilde{\sigma}^{-\frac{6}{25}} \frac{\partial F_1^\epsilon}{\partial \eta_2} \right], \tag{36}$$

where  $\boldsymbol{\eta}^{-1}$  is the inverse function of  $\boldsymbol{\eta}$ ,  $\tilde{\sigma} = \sigma \circ \boldsymbol{\eta}^{-1}$ , and  $C = c_3 c_2^2 \sigma_0^{\frac{2b}{b-1}}$  as denoted in Lemma 11. Moreover, the boundary conditions are given, for all  $(\eta_1, \eta_2) \in \partial\Omega$ , by:

$$F^\epsilon|_{\partial\Omega}(\eta_1(s, \tau), \eta_2(s, \tau)) = (u^\epsilon|_{\partial\Omega_\epsilon}(s, \tau), 0). \tag{37}$$

*Proof.* First, under these conditions,  $u^\epsilon$  and  $v^\epsilon$  verify the system described in Theorem 15, and, according to Lemma 16,  $\boldsymbol{\eta}$  is a diffeomorphism of  $\Omega_\epsilon$ . Second, Eq. (35) allows the definition of a stream-function given a fixed point  $(s_0, \tau_0) \in \Omega_\epsilon$ . Third, Eq. (36) is written in terms of its partial derivatives. Then, these partial derivatives are calculated in the new coordinates  $\eta_1$  and  $\eta_2$ . Finally, the left side and right side of the new equation for the stream-function's original partial derivatives is presented in the new directions, and substituted by the field's  $F^\epsilon$  coordinate functions. The boundary conditions are determined as a direct result of the vector field's definition, where it can be seen that it satisfies the relations: For all  $(\eta_1, \eta_2) \in \Omega$  such that  $\boldsymbol{\eta}(s, \tau) = (\eta_1, \eta_2)$ ,

$$F_1^\epsilon(\eta_1, \eta_2) = \frac{\partial\psi}{\partial\eta_2}(\eta_1, \eta_2) = \frac{1}{\rho^\epsilon} \frac{\partial\psi}{\partial\tau}(s, \tau) = u^\epsilon(s, \tau); \quad \text{and} \tag{38}$$

$$F_2^\epsilon(\eta_1, \eta_2) = -\frac{\partial\psi}{\partial\eta_1}(\eta_1, \eta_2) = -\rho^\epsilon \frac{\partial\psi}{\partial s}(s, \tau) = -(\rho^\epsilon)^2 v^\epsilon(s, \tau). \tag{39}$$

In particular, the repeated argument made for Eq. (25) and the Cauchy-Schwarz inequality for the  $L^2$ -norm implies that the vector  $F^\epsilon \in L^2(\Omega; \mathbb{R}^2)$ . Moreover,  $F^\epsilon \in L^1_{loc}(\Omega; \mathbb{R}^2)$  and has inherited locally integrable weak partial derivatives.

If Eq. (38), for each fixed point  $(s_0, \tau_0) \in \Omega_\epsilon$  and each  $(s, \tau) \in \Omega_\epsilon$ , the Poincare's Lemma implies that the integral

$$\psi(s, \tau) : = \int_\gamma (-\rho^\epsilon v^\epsilon) ds + (\rho^\epsilon u^\epsilon) d\tau,$$

has the same real value for every  $\gamma: [0, 1] \rightarrow \bar{\Omega}_\epsilon$  such that  $\gamma(0) = (s_0, \tau_0)$  and  $\gamma(1) = (s, \tau)$ . This is, the streamfunction  $\psi$  is well defined on  $\Omega_\epsilon$ .

In order to calculate its derivatives, it is enough to pick a trajectory built by pieces where one variable is fixed. Substitution of  $u^\epsilon$  and  $v^\epsilon$  in terms of the streamfunction's derivatives,  $\partial\psi/\partial s = -\rho^\epsilon v^\epsilon$  and  $\partial\psi/\partial\tau = \rho^\epsilon u^\epsilon$ , and the hypothesis that  $\rho^\epsilon$  is not null at any point of its domain, allows us to write Eq. (36) in terms of  $\psi$  as:

$$L^2\epsilon^2 \left[ \frac{\partial\psi}{\partial\tau} \frac{\partial}{\partial s} \left[ \frac{1}{\rho^\epsilon} \frac{\partial\psi}{\partial\tau} \right] - \frac{\partial\psi}{\partial s} \frac{\partial}{\partial\tau} \left[ \frac{1}{\rho^\epsilon} \frac{\partial\psi}{\partial\tau} \right] \right] = c_3 \frac{\partial}{\partial\tau} \left[ \sigma^{\frac{19}{25}} \frac{\partial}{\partial\tau} \left( \frac{1}{\rho^\epsilon} \frac{\partial\psi}{\partial\tau} \right) \right].$$

Additionally, there is a new domain  $\Omega \subset \mathbb{R}^2$  where:

$$\frac{\partial\psi}{\partial\tau} = \frac{\partial\psi}{\partial\eta_2} \frac{\partial\eta_2}{\partial\tau} = \rho^\epsilon \frac{\partial\psi}{\partial\eta_2} \quad \& \quad \frac{\partial\psi}{\partial s} = \frac{\partial\psi}{\partial\eta_1} \frac{\partial\eta_1}{\partial s} = \frac{1}{\rho^\epsilon} \frac{\partial\psi}{\partial\eta_1}. \tag{40}$$

Once again, substitution of identities in Eq. (40) in the left side of the equation above and the definition of  $F^\epsilon$  give a new expression for the nonlinear term as:

$$L^2\epsilon^2 \left[ \frac{\partial\psi}{\partial\eta_2} \frac{\partial^2\psi}{\partial\eta_1\partial\eta_2} - \frac{\partial\psi}{\partial\eta_1} \frac{\partial^2\psi}{\partial\eta_2^2} \right] = L^2\epsilon^2 \left[ F_1^\epsilon \frac{\partial F_1^\epsilon}{\partial\eta_1} + F_2^\epsilon \frac{\partial F_1^\epsilon}{\partial\eta_2} \right].$$

Similarly, by the second identity in Eq. (40) and the definition of  $F_1^\epsilon$ , the right side of the same equation is:

$$\begin{aligned} c_3 \frac{\partial}{\partial\tau} \left[ \sigma^{\frac{19}{25}} \frac{\partial}{\partial\tau} \left( \frac{1}{\rho^\epsilon} \frac{\partial\psi}{\partial\tau} \right) \right] &= c_3 \frac{\partial\eta_2}{\partial\tau} \frac{\partial}{\partial\eta_2} \left[ \sigma^{\frac{19}{25}} \rho^\epsilon \frac{\partial^2\psi}{\partial\eta_2^2} \right] \\ &= c_3 \rho^\epsilon \frac{\partial}{\partial\eta_2} \left[ \sigma^{\frac{19}{25}} \rho^\epsilon \frac{\partial^2\psi}{\partial\eta_2^2} \right] \\ &= c_3 c_1^2 \sigma_0^{\frac{2b}{(b-1)}} \tilde{\sigma}^{-1} \frac{\partial}{\partial\eta_2} \left[ \sigma^{\frac{19}{25}-1} \frac{\partial F_1^\epsilon}{\partial\eta_2} \right]. \end{aligned}$$

Therefore, the vector field  $F^\epsilon \in L^2(\Omega; \mathbb{R}^2) \cap L^1_{loc}(\Omega; \mathbb{R}^2)$ , and its locally integrable weak partial derivatives, satisfy the incompressible system of Eq. (35) and (36) with boundary conditions given by Eq. (37).  $\square$

### 2.3. Dorodnitzyn Boundary Layer Limit Formula

Alberto Bressan's [5] book *Lecture Notes on Functional Analysis: With Applications to Linear Partial Differential Equations* provides an excellent account of Sobolev Embedding Theorems, as they will be used in this section.



**Theorem 18.** *Under the same hypothesis of Theorem 17, there is an estimate:*

$$\|\nabla F^\epsilon\|_{L^2(\Omega; \mathbb{R}^2)} \leq \frac{c_2 U^3}{2C}. \tag{41}$$

*Proof.* From Theorem 17, the vector field  $F^\epsilon = (F_1^\epsilon, F_2^\epsilon)$  verifies the system of Eq. (35) and (36) in  $\Omega$  with boundary conditions determined by (37). In particular, there is an underlying assumption that the Laplacian

$$\Delta F_2^\epsilon = \sum_{i=1,2} \frac{\partial^2 F_2^\epsilon}{\partial \eta_i^2} = 0, \tag{42}$$

because the conservation of momentum equation for  $F_2^\epsilon$  is considered null. Furthermore, from  $\partial u / \partial x = 0$  a.e. in  $\Omega_h$  and Eq. (38), it can be seen that:

$$\frac{\partial^2 F_1^\epsilon}{\partial \eta_1^2} = 0. \tag{43}$$

Let  $\mathcal{F}_1(F^\epsilon)$  denote the inner product in  $L^2(\Omega; \mathbb{R}^2)$  of  $F^\epsilon$  and the vector  $(F^\epsilon \cdot \nabla) F^\epsilon = \left(\sum_{i=1,2} F_i^\epsilon \frac{\partial F_j^\epsilon}{\partial \eta_i}\right)_{j=1,2}$  in the space  $L^2(\Omega; \mathbb{R}^2)$ . Namely,

$$\mathcal{F}_1(F^\epsilon) = \frac{1}{2} \iint_{\Omega} \sum_{i=1,2} F_i^\epsilon \left( \sum_{j=1,2} \frac{\partial (F_j^\epsilon)^2}{\partial \eta_i} \right) d\eta.$$

From Eq. (38) and the boundary conditions (29), (30), and (31) for  $v^\epsilon$ , we have:

$$F_2^\epsilon|_{\partial\Omega} = 0. \tag{44}$$

If  $\text{div}(F_1^\epsilon, F_2^\epsilon) = 0$ , by the Gauss-Ostrogradsky Theorem and Eq. (44):

$$\begin{aligned} \mathcal{F}_1(F^\epsilon) &= -\frac{1}{2} \iint_{\Omega} \left\{ (F_1^\epsilon)^2 + (F_2^\epsilon)^2 \right\} \text{div}(F_1^\epsilon, F_2^\epsilon) d\eta \\ &\quad + \frac{1}{2} \int_{\partial\Omega} \left( (F_1^\epsilon)^2, 0 \right) \cdot \mathbf{n} dS, \\ &= \frac{1}{2} \int_{\partial\Omega} \left( (F_1^\epsilon)^3, 0 \right) \cdot \mathbf{n} dS, \end{aligned}$$

where  $\mathbf{n}$  is the outward pointing unitary normal vector field of the topological boundary  $\partial\Omega$ . Because  $\eta$  is a diffeomorphism,  $\eta(\partial\Omega_\epsilon) = \partial\Omega$ . This is,

$\partial\Omega = \boldsymbol{\eta}(\phi^\epsilon(\Gamma_0)) \cup \boldsymbol{\eta}(\phi^\epsilon(\Lambda_0)) \cup \boldsymbol{\eta}(\phi^\epsilon(\Lambda_L)) \cup \boldsymbol{\eta}(\phi^\epsilon(\Gamma_h))$ . The no-slip boundary condition for  $F_1^\epsilon$  in  $\boldsymbol{\eta}(\phi^\epsilon(\Gamma_0))$  is inherited from  $u^\epsilon$  by Eq. (29). In this way,

$$\int_{\boldsymbol{\eta}(\phi^\epsilon(\Gamma_0))} \left( (F_1^\epsilon)^3, 0 \right) \cdot \mathbf{n} \, dS = 0.$$

The periodic boundary conditions of  $u^\epsilon$  established in Eq. (31) imply that  $\forall \tau \in [0, \delta/L\epsilon]$ , or for all  $(1, \tau) \in \boldsymbol{\eta}(\phi^\epsilon(\Lambda_L))$ :

$$\begin{aligned} \eta_1(1, \tau) &= c_2^{-1} \sigma_0^{-b/(b-1)} \left( 1 - \left[ u^\epsilon(1, \tau)^2 - u^\epsilon(0, \tau)^2 \right] \right) \\ &= c_2^{-1} \sigma_0^{-b/(b-1)}. \end{aligned}$$

In addition,  $\eta_1(0, \tau) = 0 \, \forall \tau \in [0, \delta/L\epsilon]$ , *i.e.*  $\forall (0, \tau) \in \boldsymbol{\eta}(\phi^\epsilon(\Lambda_0))$ . Thus, the partial derivatives  $\partial\eta_1/\partial\tau(0, \tau) = \partial\eta_1/\partial\tau(1, \tau) = 0 \, \forall \tau \in [0, \delta/L\epsilon]$ , and the boundary's sections  $\boldsymbol{\eta}(\phi^\epsilon(\Lambda_0))$  and  $\boldsymbol{\eta}(\phi^\epsilon(\Lambda_L))$  are vertical. Consequently, Eq. (38) implies that:

$$\int_{\boldsymbol{\eta}(\phi^\epsilon(\Lambda_0))} \left( (F_1^\epsilon)^3, 0 \right) \cdot \mathbf{n} \, dS = - \int_0^{\delta/L\epsilon} [u^\epsilon(0, \tau)]^3 \frac{\partial\eta_1}{\partial\tau}(0, \tau) \, d\tau = 0.$$

Similarly,

$$\int_{\boldsymbol{\eta}(\phi^\epsilon(\Lambda_L))} \left( (F_1^\epsilon)^3, 0 \right) \cdot \mathbf{n} \, dS = 0.$$

As a result, the product  $\mathcal{F}_1(F^\epsilon)$  is determined only by the free-stream velocity:

$$\mathcal{F}_1(F^\epsilon) = \frac{1}{2} \int_{\boldsymbol{\eta}(\phi^\epsilon(\Gamma_h))} ([-LU^\epsilon]^3, 0) \cdot \mathbf{n} \, dS. \tag{45}$$

Let  $\mathcal{F}_2(F^\epsilon)$  designate the product of  $F^\epsilon$  and the vector corresponding to the right side of Eq. (36) in the space  $L^2(\Omega; \mathbb{R}^2)$ :

$$\begin{aligned} \mathcal{F}_2(F^\epsilon) &= \iint_{\Omega} (F_1^\epsilon, F_2^\epsilon) \cdot \left( C\tilde{\sigma}^{-1} \frac{\partial}{\partial\eta_2} \left[ \tilde{\sigma}^{-\frac{6}{25}} \frac{\partial F_1^\epsilon}{\partial\eta_2} \right], 0 \right) \, d\boldsymbol{\eta}, \\ &= C \iint_{\Omega} F_1^\epsilon \tilde{\sigma}^{-1} \frac{\partial}{\partial\eta_2} \left[ \tilde{\sigma}^{-\frac{6}{25}} \frac{\partial F_1^\epsilon}{\partial\eta_2} \right] \, d\boldsymbol{\eta}. \end{aligned}$$

In fact,  $\tilde{\sigma}^{-1} > 1$  in  $\bar{\Omega}$ . Then, by the Gauss-Ostrogradsky Theorem, Eq. (38), and the boundary conditions (29), (30) and (31), we have:

$$\mathcal{F}_2(F^\epsilon) \geq C \iint_{\Omega} F_1^\epsilon \frac{\partial}{\partial\eta_2} \left[ \tilde{\sigma}^{-\frac{6}{25}} \frac{\partial F_1^\epsilon}{\partial\eta_2} \right] \, d\boldsymbol{\eta},$$

$$\begin{aligned} &\geq -C \left[ \iint_{\Omega} \left( \frac{\partial F_1^\epsilon}{\partial \eta_2} \right)^2 d\boldsymbol{\eta} + \frac{1}{2} \int_{\partial\Omega} \left( 0, \frac{\partial (F_1^\epsilon)^2}{\partial \eta_2} \right) \cdot \mathbf{n} dS \right] \\ &= -C \iint_{\Omega} \left( \frac{\partial F_1^\epsilon}{\partial \eta_2} \right)^2 d\boldsymbol{\eta}. \end{aligned}$$

This is because the restriction of  $F_1^\epsilon$  to  $\boldsymbol{\eta}(\phi^\epsilon(\Gamma_h))$  is constant, the derivative  $\partial F_1^\epsilon / \partial \eta_2|_{\boldsymbol{\eta}(\phi^\epsilon(\Gamma_h))} = 0$ , and the periodic boundary condition (31) makes vertical the sections  $\boldsymbol{\eta}(\phi^\epsilon(\Lambda_0))$  and  $\boldsymbol{\eta}(\phi^\epsilon(\Lambda_L))$ , so that the normal  $\mathbf{n}|_{\boldsymbol{\eta}(\phi^\epsilon(\Lambda_0)) \cup \boldsymbol{\eta}(\phi^\epsilon(\Lambda_L))} = (\pm 1, 0)$ .

In similar fashion, given Eq. (43) and  $\partial F_1^\epsilon / \partial \eta_1|_{\boldsymbol{\eta}(\phi^\epsilon(\Gamma_h))} = 0$ :

$$\begin{aligned} \iint_{\Omega} \left( \frac{\partial F_1^\epsilon}{\partial \eta_1} \right)^2 d\boldsymbol{\eta} &= - \iint_{\Omega} F_1^\epsilon \frac{\partial^2 F_1^\epsilon}{\partial \eta_1^2} d\boldsymbol{\eta} \\ &\quad + \int_{\boldsymbol{\eta}(\phi^\epsilon(\Gamma_h))} \left( F_1^\epsilon \frac{\partial F_1^\epsilon}{\partial \eta_1}, 0 \right) \cdot \mathbf{n} dS = 0. \end{aligned}$$

And, in the same way, Eq. (42) and (44) imply that:

$$\sum_{i=1,2} \iint_{\Omega} \left( \frac{\partial F_2^\epsilon}{\partial \eta_i} \right)^2 d\boldsymbol{\eta} = - \iint_{\Omega} F_2^\epsilon \Delta F_2 d\boldsymbol{\eta} = 0.$$

Therefore, if Eq. (36) is satisfied by  $F^\epsilon$ , then  $\mathcal{F}_1(F^\epsilon) = \mathcal{F}_2(F^\epsilon)$ , and Eq. (45) gives:

$$\begin{aligned} \|\nabla F^\epsilon\|_{L^2(\Omega; \mathbb{R}^2)} &= \iint_{\Omega} \left( \frac{\partial F_1^\epsilon}{\partial \eta_2} \right)^2 d\boldsymbol{\eta} \\ &\leq \frac{1}{2C} \int_{\boldsymbol{\eta}(\phi^\epsilon(\Gamma_h))} ([LU^\epsilon]^3, 0) \cdot \mathbf{n} dS. \end{aligned}$$

Finally, each density value  $\rho = \rho^\epsilon \leq c_2$ , and  $\partial \eta_1 / \partial s = \rho^\epsilon$  in  $\Omega$ . Hence,

$$\begin{aligned} \frac{1}{2C} \int_{\boldsymbol{\eta}(\phi^\epsilon(\Gamma_h))} ([LU^\epsilon]^3, 0) \cdot \mathbf{n} dS &\leq \frac{U^3}{2C} \int_0^1 \frac{\partial \eta_1}{\partial s}(s, h^\epsilon(s)) ds \\ &\leq \frac{c_2 U^3}{2C}. \end{aligned}$$

□

**Theorem 19.** *Without loss of generality, assume  $L, H > 1$ . Under the same hypothesis of Theorem 17, and the additional existence of locally integrable generalized derivatives up to order 2 for  $u$ , we obtain that  $u$  is a weak solution to the limit formula:*

$$f \frac{\partial^2 u}{\partial y^2} = \frac{\partial f}{\partial y} \frac{\partial u}{\partial y}, \tag{46}$$

in  $L^2(\Omega_h; \mathbb{R}^2)$ , where  $f = [1 - (u^2(x, y) / 2i_0)]^{-\frac{6}{25}}$ .

*Proof.* If  $\mathbf{v} = (u, v) \in L^2(\Omega_h; \mathbb{R}^2)$ , and  $L^2\epsilon = LH > 1$ :

$$\begin{aligned} \iint_{\Omega} (F_1^\epsilon(\eta_1, \eta_2))^2 d\eta_1 d\eta_2 &= \iint_{\Omega_\epsilon} (u^\epsilon(s, \tau))^2 ds d\tau \\ &= LH \iint_{\Omega_h} (u(x, y))^2 dx dy \\ &= LH \|u\|_{L^2(\Omega_h)}^2. \end{aligned}$$

In a similar manner, the estimate  $\rho^\epsilon \leq c_2$  implies that:

$$\begin{aligned} \iint_{\Omega} (F_2^\epsilon(\eta_1, \eta_2))^2 d\eta_1 d\eta_2 &= \iint_{\Omega_\epsilon} ((\rho^\epsilon)^2 v^\epsilon(s, \tau))^2 ds d\tau \\ &\leq c_2^4 LH \|v\|_{L^2(\Omega_h)}^2. \end{aligned}$$

Therefore,

$$\|F^\epsilon\|_{W^{1,2}(\Omega; \mathbb{R}^2)}^2 \leq L \left\{ H \|u\|_{L^2(\Omega_h)}^2 + c_2^4 H \|v\|_{L^2(\Omega_h)}^2 + \frac{c_2 U^3}{2C} \right\}.$$

Thus, the sequence  $(F^\epsilon)$  is contained and bounded in the Sobolev Space  $W^{1,2}(\Omega; \mathbb{R}^2)$  by a constant value independent of the parameter  $\epsilon > 0$ . As a consequence, the Rellich-Kondrachov compactness theorem, A. Bressan [5, p. 173, 178], implies that it has a subsequence  $(F^{\epsilon_\alpha})$  that converges strongly in  $L^2(\Omega; \mathbb{R}^2)$ , and the sequence  $\partial F_1^\epsilon / \partial \eta_2$  converges weakly in  $L^2(\Omega)$  to the generalized derivative  $\partial F_1 / \partial \eta_2$  of the limit  $F = (F_1, F_2) \in L^2(\Omega; \mathbb{R}^2)$ . But,  $F_1^\epsilon = u^\epsilon = 1/L u$  for all  $\epsilon > 0$ . Then, the horizontal velocity  $u$  is a weak solution of the limit formula, Eq. (46), in  $L^2(\Omega_h)$  when the parameter  $\epsilon$  tends to 0. □

### 3. Conclusion

The obtained limit formula suggests that there is no separation of the Boundary Layer under this conditions, but it shows that it is possible to study the change of the horizontal velocity of atmospheric wind with height near the surface by means of simpler models. There are two immediate problems to work on: First, to obtain solutions to the Reynolds' limit model by the application of fractional calculus methods. Second, to consider the case where the Neumann condition  $\partial T/\partial z|_{z=0} = m$  is a constant  $m \neq 0$ .

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