

**DORODNITZYN'S SHEAR STRESS  
REYNOLDS' LIMIT FORMULA**

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**Abstract:** A previous analysis by the author (published previously in this Journal) showed that a limit formula could be deduced from Dorodnitsyn's compressible boundary layer model by the application of Bayada and Chambat's diffeomorphism. This article is the second part of the same research. Now, a limit formula in terms of the shear stress is deduced from Dorodnitsyn's shear stress model.

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**Key Words:** boundary-layer theory; Navier-Stokes Equations; Reynold's limit formulas

**1. Introduction**

Theorem 4 presents a proof of Dorodnitsyn's derivation of a shear stress equation in a rectangular domain in terms of each variable's belonging to a specific functional space. Theorem 5 gives a new limit model in terms of the shear stress.

We may recall that Dorodnitsyn reduced the original system of seven equations for seven variables to a quasi-linear problem for a transformation of the shear stress in a new domain. Surely, there exists a mathematical formalization preceding from the one given here, but the author could not find it in the

literature. This might be a consequence of the fact that Dorodnitsyn's work of the subsequent years is *partially classified* [1, p.1973].

The proportion  $L \gg \gg h > 0$  allows the introduction of a small parameter  $\epsilon = h/L$  and the application of Bayada's change of variables [2] to obtain a Reynolds' Limit Formula. Theorem 5 gives a demonstration of a Reynolds' Limit Formula for Dorodnitsyn's shear stress quasi-linear problem.

## 2. Problem Statement

From this point forward,  $W^{k,p}(D)$  denotes the Sobolev Space of elements in the Lebesgue Space  $L^p(D)$  on a domain  $D \subset \mathbb{R}^2$  with generalized derivatives up to the order  $k$ , all of which belong to  $L^p(D)$ . We might recall that [3], [4]:

**Definition 1.** A *domain* is an open and connected subset  $D \subset \mathbb{R}^2$  of the Euclidean space  $\mathbb{R}^2$ . A distribution  $g \in L^1(D)$  is a *generalized derivative* of  $f$  with respect to  $x$  – also called *weak* or *distributional*, if for all analytic functions  $\varphi$  with compact support in  $D$ ,  $\varphi \in C_0^\infty(D)$ , we have:

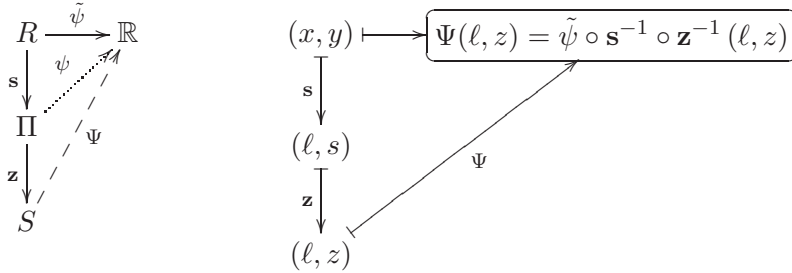
$$\iint_D f \frac{\partial \varphi}{\partial x} dx dy = - \iint_D g \varphi dx dy.$$

Analogously, it can be defined for other coordinate systems and orders. A necessary and sufficient condition for the density of  $C^\infty(\bar{D})$  in a Sobolev Space  $W^{k,2}(D)$  is unknown [4, p.10]. However, it is enough for the domain  $D$  to be a rectangle. Therefore, the following results can be stated for a  $\hat{f} \in C^\infty(\bar{D})$  approximation of each distribution  $f \in W^{k,p}(D)$ .

As a particular case, Leibnitz Rule for product differentiation is valid in a non-empty open domain  $D \subset \mathbb{R}^2$  when both factors and all the generalized derivatives involved are elements of  $L^2(R)$  [3, p.11]. Moreover, there is a generalized Green's Theorem [5, p. 121] that is valid for elements of the Sobolev Spaces  $W^{1,2}(D)$  in a bounded *Lipschitz domain* ([4])  $D \subset \mathbb{R}^2$ . This allows the existence of a *stream function*, defined in Theorem 3.

The quasi-linear statement of the original problem in terms of the shear stress is obtained by a series of two essential steps. First, Theorem 3 shows that the original problem has a simplified expression as a system of just one condition for the *stream function*  $\psi$  taken over the polygon  $\Pi = \mathbf{s}(R)$  in terms of Dorodnitsyn's change of coordinates  $\mathbf{s}(x, y) = (\ell, s)$  of the original rectangular domain  $R$ , where the convective derivative has an incompressible form.

Second, Theorem 4 gives a formal proof of how this system can be written in terms of a transformation that takes the original shear stress to a new domain, an infinite strip band  $S := \{(\ell, z) \in \mathbb{R}^2 \mid (\ell, s) \in \Pi, \text{ and } z = s/\ell^{1/2} \in (0, \infty)\}$ , following a composition of the original *stream function* with Dorodnityzn's diffeomorphism  $\mathbf{s}$ , and with Blasius' adapted height normalization  $z = s/\ell^{1/2}$ , [6]:



**Definition 2.** Let  $L \gg \gg h > 0$ ,  $R = [0, L] \times [0, h]$  and  $\hat{R} = R \times [0, h]$ . If  $\hat{\rho} \in L^1(\hat{R} \times [0, \infty); (0, \infty))$  such that  $\partial \hat{\rho} / \partial t = 0$ , then the restriction  $\rho = \hat{\rho}|_R$ ,  $\rho \in L^2(R; (0, \infty))$ ; a horizontal velocity component  $u \in L^2(R)$  with generalized derivatives  $\partial u / \partial x, \partial u / \partial y, \partial^2 u / \partial y^2 \in L^2(R)$ ; a vertical velocity component  $v \in L^2(R)$ ; an absolute temperature  $T \in L^2(R; (0, \infty))$  such that  $\partial T / \partial y, \partial^2 T / \partial y^2 \in L^2(R)$ ; a dynamic viscosity  $\mu \in L^2(R)$ ; a pressure  $p \in L^2(R)$ ; and a thermal conductivity  $\kappa \in L^2(R)$ . Additionally, assume that both products  $\rho u, \rho v \in L^2(R)$ , and that all of them have first order generalized derivatives in  $L^2(R)$ . This is,  $\rho, u, v, T, \mu, p$  and  $\kappa$  are elements of the space  $W^{1,2}(R)$ .

We need to retrieve not only Dorodnityzn's stationary gaseous boundary layer model with constant total energy [7] in order to give a formal proof of his shear stress statement of the problem and deduce its corresponding limit formula, but also to provide the functional spaces where this demonstration holds.

These are, Eqs. (1), (2), (3), (4), (5), (6), (7), and boundary conditions – Eq. (8), (9), (10), (11), (13), (12), in a long rectangle  $R = (0, L) \times (0, h) \in \mathbb{R}^2$  that represent the boundary layer region for  $L \gg \gg h > 0$ . Dorodnityzn's model is based on three simplified stationary Conservation of Mass, Conservation of Momentum, and Conservation of Energy laws, Eqs. (1), (2) and (3),

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0, \tag{1}$$

$$\rho \left( u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = - \frac{\partial p}{\partial x} + \frac{\partial}{\partial y} \left( \mu \frac{\partial u}{\partial y} \right), \quad y \quad (2)$$

$$\rho \left[ u \frac{\partial (c_p T)}{\partial x} + v \frac{\partial (c_p T)}{\partial y} \right] = \frac{\partial}{\partial y} \left[ \kappa \frac{\partial T}{\partial y} \right] + \mu \left( \frac{\partial u}{\partial y} \right)^2 + \frac{\partial p}{\partial t}, \quad (3)$$

for a *stationary density*  $\rho$ , a horizontal and vertical *velocity components*,  $u$  and  $v$ , an *absolute temperature*  $T$ , a *dynamic viscosity*  $\mu$ , a *pressure*  $p$ , and a *thermal conductivity*  $\kappa$  whose main assumptions as elements of the Lebesgue space  $L^2(R)$  are described in Definition 2. Under these assumptions, the complete system is made up of seven identities in the Lebesgue space  $L^1(R)$ .

The value  $c_p$  is the *specific heat at constant pressure*. It is worth to notice that there is a considerable difference between values of a *gas constant for dry air*  $\hat{R}_d = 287 [J K^{-1}kg^{-1}]$ , and a *gas constant for saturated water vapor* [8, p.1047]  $\hat{R}_v = 461.50 [J K^{-1}kg^{-1}]$ ; the *specific heat at constant pressure for dry air* [9]  $c_{pd} = 1004 [JK^{-1}kg^{-1}]$  and the *specific heat at constant pressure for water vapor* [8]  $c_{pv} = 1875 [JK^{-1}kg^{-1}]$ . Therefore, one question that arises is if each model's solution will continuously vary with modifications of these constants and what consequences does it have on the boundary layer separation.

Furthermore, we have four Ideal Gas Thermodynamic Laws, Eq. (4), (5), (6), (7): the *Prandtl number*  $Pr = 1$ ,

$$Pr = \frac{c_p \mu}{\kappa} = 1; \quad (4)$$

the *Equation of State* for the *Universal Gas constant*  $R^*$ , the volume  $V$  of a rectangular prism  $[0, L] \times [0, h] \times [0, h] \subset \mathbb{R}^3$  and the *number of moles*  $n$  of an *ideal gas* corresponding to the volume  $V$ ,

$$pV = nR^*T; \quad (5)$$

the *adiabatic polytropic atmosphere* [10, p. 35] where  $b = 1.405$  and  $c$  are constants,

$$pV^b = c; \quad (6)$$

and the *Power-Law* [11, p. 46]

$$\frac{\mu}{\mu_h} = \left( \frac{T}{T_h} \right)^{\frac{19}{25}}, \quad (7)$$

with boundary conditions, Eqs. (8), (9), (10), (11), (13), (12):

$$(u, v)|_{\{(x,h): 0 \leq x \leq L\}} = (-U, 0), \quad (8)$$

$$(u, v)|_{\{(x,0): 0 \leq x \leq L\}} = (0, 0), \tag{9}$$

for a positive value of the *free-stream velocity*,  $U > 0$ , the *no slip condition* at the surface, a *free-stream temperature*  $T_h > 0$ , a *free-stream dynamic viscosity*  $\mu_h > 0$ ,

$$T|_{\{(x,h): 0 \leq x \leq L\}} = T_h > 0, \tag{10}$$

$$\mu|_{\{(x,h): 0 \leq x \leq L\}} = \mu_h > 0, \tag{11}$$

and a Neumann condition:

$$\frac{\partial T}{\partial y} \Big|_{\{(x,0): 0 \leq x \leq L\}} = 0. \tag{12}$$

In [12], periodic conditions, such as the ones used in Chupin and Sart's work [13], were included at the vertical sections of the *topological boundary*  $\partial R$ , such that for all  $y \in [0, h]$ :

$$(u(0, y), 0) = (u(L, y), 0). \tag{13}$$

Luigi Crocco's Procedure, described in the original article [14], can be applied to the distributions  $\rho, u, v, T, p, \kappa, \mu$  because the generalized derivatives of the variables are elements of the Lebesgue space  $L^2(R)$ , and we can proceed as we would with classical derivatives to apply a generalized Leibnitz Rule for the product [3, p.11], so that Eq. (3) is satisfied if and only if:

$$\rho \left[ u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} \right] \left( c_p T + \frac{u^2}{2} \right) = \frac{\partial}{\partial y} \left[ \mu \frac{\partial}{\partial y} \left( c_p T + \frac{u^2}{2} \right) \right]. \tag{14}$$

Moreover,  $T(u) = T_0 (1 - u^2/(2c_p T_0))$  where

$$T_0 = T_h + 1 - (U^2/2c_p) > 0,$$

and  $i_0 = c_p T_0 > 0$ . If we take into account the *atmospheric pressure* expression  $p(x, \hat{y}) = g \int_{\hat{y}}^{\infty} \rho(x, y) dy$  for the *standard gravity*  $g$  and a linear decrease  $T(x, y) = T_0 - \beta y$  for a constant  $\beta > 0$  for  $(x, y) \in R$ , then:

$$p \cong c_1 [1 - (U^2/2i_0)]^{\frac{b}{(b-1)}},$$

and the density  $\rho(u) \cong c_2 [1 - (U^2/2i_0)]^{\frac{b}{(b-1)}} / [1 - (u^2(x, y)/2i_0)]$ . From Eq. (5), the dynamic viscosity  $\mu(u) = c_3 [1 - (u^2/2i_0)]^{\frac{19}{25}}$  for a gas constant  $\hat{R} = R^*/M$ , the molecular weight  $M$ ,  $p_0 = (n R^* T_0) / V > 0$ ,  $c_1 = p_0 T_0^{\frac{2b}{b-1}}$ ,  $c_2 = c_1 \hat{R}^{-1} T_0^{-1}$ ,  $c_3 = \mu_h T_h^{-\frac{19}{25}} T_0^{\frac{19}{25}}$ .

**Theorem 3.** Let  $\rho, u, v, T, p, \kappa, \mu$  be as in Definition 2. Assume  $p = c_1 [1 - (U^2/2i_0)]^{\frac{b}{(b-1)}}$ ,  $\partial u/\partial x = 0$ , and that the variables verify Eq. (1), (2), (14), (4), (5), (6), (7) and (8), (9), (10), (11), (13), (12). Consider  $R \xrightarrow{\mathbf{s}} \Pi$ ,  $(x, y) \xrightarrow{\mathbf{s}} (\ell, s)$ , where:

$$\ell(\hat{x}, \hat{y}) \longmapsto \int_0^{\hat{x}} p(x, \hat{y}) dx$$

and

$$s(\hat{x}, \hat{y}) \longmapsto \int_0^{\hat{y}} \rho(\hat{x}, y) dy.$$

Denote  $\Pi = \mathbf{s}(R)$  and  $\sigma_0 = 1 - U^2/(2i_0)$ . Then, there is a stream-function  $\tilde{\psi} \in W^{2,2}(R)$  such that  $\partial \tilde{\psi}/\partial x = -\rho v$ ,  $\partial \tilde{\psi}/\partial y = \rho u$ , and a  $\tilde{\sigma} = 1 - (u^2/2i_0) \in W^{1,2}(R; (0, \infty))$ , such that  $\psi := \tilde{\psi} \circ \mathbf{s}^{-1} \in W^{2,2}(\Pi)$  and  $\sigma := \tilde{\sigma} \circ \mathbf{s}^{-1} \in W^{1,2}(\Pi; (0, \infty))$  satisfy:

$$\frac{\partial \psi}{\partial s} \frac{\partial^2 \psi}{\partial \ell \partial s} - \frac{\partial \psi}{\partial \ell} \frac{\partial^2 \psi}{\partial s^2} = c_1^{-1} c_2 c_3 \sigma_0^{\frac{b}{(b-1)}-1} \frac{\partial}{\partial s} \left[ \sigma^{\frac{19}{25}-1} \frac{\partial^2 \psi}{\partial s^2} \right]. \tag{15}$$

*Proof.* First, we describe Dorodnitzyn’s diffeomorphism: The new domain’s,  $\Pi$ , extremes are  $\ell_M = \ell(0, L) = c_1 \sigma_0 L$  and  $s(x, h) = c_2 \sigma_0^{\frac{b}{(b-1)}-1} h$ . The partial derivatives of  $\ell$  over  $R$  are  $\partial \ell/\partial x = c_1 \sigma_0$  and  $\partial \ell/\partial y = 0$ . Given that  $\partial u/\partial x = 0$ , the *Dominated Convergence Theorem* [15, p.44] implies that  $\partial s/\partial x = 0$ . Moreover,  $\partial s/\partial y = \rho$ . This may clarify the definition of  $s$  as the *entropy* [16, p.432] and Dorodnitzyn’s statment of the problem as an *entropy method*.

The *Jacobian determinant*  $|Ds| = c_1 \sigma_0 \rho > 0$ . Thus, the *Inverse Function Theorem* [17] implies that  $\mathbf{s}$  is a *diffeomorphism* that takes the rectangle  $R$  into a polygonal domain  $\Pi$ . In this coordinate system, *von Kármán’s Integral Formula* for a compressible fluid in  $R$  has an incompressible form in  $\Pi$  [18, p.258].

Because of the zero divergence given in Eq. (1), the generalized Green Theorem for Sobolev Spaces  $W^{1,2}(R)$  on a rectangular domain  $R$  [5, p.121] and the Poincaré Lemma allow us to define a *stream function*  $\tilde{\psi} \in W^{2,2}(R)$ ,  $\tilde{\psi}_{(0,0)}(x, y) = \int_{(0,0)}^{(x,y)} -\rho v dx$ . The *stream function*  $\tilde{\psi}$  is regarded in  $\Pi$  as  $\psi \in W^{2,2}(\Pi)$ . Once more, over the rectangular domain  $R$ , we can apply the Leibniz Rule for  $L^2$ -distributions to see that, in terms of  $\psi$ , the system has an incompressible non-linear expression for the convective derivative term in the left hand side of Eq. (2) in  $\Pi$  as:

$$\rho \left( u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = c_1 \sigma_0 \rho \left( \frac{\partial \psi}{\partial s} \frac{\partial^2 \psi}{\partial \ell \partial s} - \frac{\partial \psi}{\partial \ell} \frac{\partial^2 \psi}{\partial s^2} \right).$$

This way, it is possible to cancel the density  $\rho$  factor with its correspondent right hand side of Eq. (2) written in  $\Pi$  as:

$$\frac{\partial}{\partial y} \left( \mu \frac{\partial u}{\partial y} \right) = c_2 c_3 \sigma_0^{\frac{b}{b-1}} \rho \frac{\partial}{\partial s} \left[ \sigma^{\frac{19}{25}-1} \frac{\partial^2 \psi}{\partial s^2} \right],$$

where  $\partial p/\partial x = 0$  because  $p$  is constant in  $R$ , and  $\sigma$  quantifies the amount of kinetic energy is transformed into heat [18]. As a distribution,  $\tilde{\sigma}(u) \in W^{1,2}(R; (0, \infty))$  and  $\partial^2 \tilde{\sigma}/\partial y^2 \in L^2(R)$  directly from  $T \in W^{1,2}(R)$  and  $\partial^2 T/\partial y^2 \in L^2(R)$ . Therefore, under the hypothesis of Definition 2 over the variables, the original problem of Eq. (1), (2), (3), (4), (5), (6), (7), is transformed into the condition given by Eq. (15) with inherited boundary conditions.  $\square$

### 3. Dorodnitzyn’s shear stress problem

At this point, Dorodnitzyn adapts Blasius’ normalization  $\mathbf{z}$  to express Eq. (15) as an the Ordinary Differential Eq. (17), which he transforms into the Quasi-Linear Parabolic Eq. (16).

**Theorem 4.** *Under the same hypotheses of Theorem 3, let*

$$S = \left\{ (\ell, z) \in \mathbb{R}^2 \mid (\ell, s) \in \Pi \text{ and } z = s/\ell^{1/2} \in (0, \infty) \right\},$$

$\Pi \xrightarrow{\mathbf{z}} S, (\ell, s) \xrightarrow{\mathbf{z}} (\ell, z), z(\ell, s) \xrightarrow{\mathbf{z}} s/\sqrt{\ell}, \Psi: = \tilde{\psi} \circ \mathbf{s}^{-1} \circ \mathbf{z}^{-1} \in W^{4,1}(S)$   
 such that  $\tilde{\Psi} = f(z) g(\ell), u_s: = u \circ \mathbf{s}^{-1} \circ \mathbf{z}^{-1}$ , and

$$\tau_s: = (1 - u_s^2/(2i_0))^{-6/25} \partial^2 f/\partial z^2.$$

Then,

$$\tau_s \frac{\partial^2 \tau_s}{\partial u_s^2} = -\frac{1}{2} c_1 c_2^{-1} c_3^{-1} \sigma_0^{1-\frac{b}{(b-1)}} u_s \left( 1 - \frac{u_s^2}{2i_0} \right)^{-6/25}. \tag{16}$$

*Proof.* First of all, the *Jacobian determinant*  $|D\mathbf{z}| = \ell^{-1/2} > 0$  for all  $\ell > 0$ . Therefore,  $\mathbf{z}$  is a *diffeomorphism* from  $\Pi$  to  $S$ . Suppose  $\Psi = g \cdot f$  is separable as the product of two distributions, independently determined by the variables  $\ell$  and  $z$ , such that  $\partial \Psi/\partial z = \ell^{1/2} \partial f/\partial z$ . Then, the Leibniz Rule for a product [19, p. 149] of  $g \in C^1(S)$  and an integrable distribution  $f \in L^1_{loc}(S)$  over an

open set  $S \neq \emptyset$ , applied to  $\partial(g \cdot f)/\partial z$  and the condition  $\partial\Psi/\partial z = \ell^{1/2} \partial f/\partial z$  imply that  $g(\ell) = \ell^{1/2}$ .

Second, if  $\psi \in W^{2,2}(\Pi)$  is a weak solution of Eq.(15), then  $f \in W^{4,1}(S)$ , such that  $\partial^k \Psi/\partial z^k = \ell^{1/2} \partial^k f/\partial z^k$  for  $k \in \{1, 2, 3, 4\}$ , is a weak solution to the ordinary differential equation:

$$-\frac{1}{2} f \frac{\partial^2 f}{\partial z^2} = c_1^{-1} c_2 c_3 \sigma_0^{\frac{b}{(b-1)}-1} \frac{\partial}{\partial z} \left( \sigma_s^{-\frac{6}{25}} \frac{\partial^2 f}{\partial z^2} \right), \tag{17}$$

where  $\sigma_s = \sigma \circ \mathbf{s}^{-1} \circ \mathbf{z}^{-1}$ . In order to verify this, we write the left and right side of Eq.(15) in terms of the new coordinates. The left side becomes:

$$\frac{\partial \psi}{\partial s} \frac{\partial^2 \psi}{\partial \ell \partial s} - \frac{\partial \psi}{\partial \ell} \frac{\partial^2 \psi}{\partial s^2} = -\frac{1}{2} \ell^{-1} f \frac{\partial^2 f}{\partial z^2}; \tag{18}$$

and, the right side is:

$$\frac{\partial}{\partial s} \left( \sigma^{-6/25} \frac{\partial^2 \psi}{\partial s^2} \right) \frac{\partial}{\partial s^2} = \ell^{-1} \frac{\partial}{\partial z} \left( \sigma^{-6/25} \frac{\partial^2 f}{\partial z^2} \right). \tag{19}$$

This way, the factor  $\ell^{-1}$  is nullified when Eq. (18) is equal to Eq. (19) and we obtain Eq. (17).

Third, let  $u_s = u \circ \mathbf{s}^{-1} \circ \mathbf{z}^{-1}$ , then  $f(z) = \int_0^z u_s(l, z') dz'$ : From the *stream-function's* separation of the first step, we have  $\partial f/\partial z = \ell^{-1/2} \partial\Psi/\partial z$ . Moreover, if  $f \in W^{1,1}(0, \infty)$ , then  $f(z) = f(0) + \int_0^z \partial f/\partial z(z') dz'$ . Because of  $\tilde{\psi}(0,0) = 0$ ,  $\psi(0,0) = \Psi(0,0) = f(0) = 0$  and  $f(z) = \int_0^z \partial f/\partial z(z') dz'$ . In addition, for each  $(\ell, z) \in S$ :  $\partial\Psi/\partial z(\ell, z) = \ell^{1/2} \partial\psi/\partial s(\mathbf{z}^{-1}(\ell, z)) = \ell^{1/2} (1/\rho) \partial\tilde{\psi}/\partial y(\mathbf{s}^{-1}(\mathbf{z}^{-1}(\ell, z))) = \ell^{1/2} u \circ \mathbf{s}^{-1} \circ \mathbf{z}^{-1}(\ell, z)$ . This is,  $\partial\Psi/\partial z(\ell, z) = \ell^{1/2} u_s(\ell, z)$ .

As a direct consequence of both relations,  $f(z) = \int_0^z u_s(l, z') dz'$  and  $\partial u_s/\partial z = \partial^2 f/\partial z^2$  Finally, if  $\tau_s = (1 - u_s^2/(2i_0))^{-6/25} \partial^2 f/\partial z^2$ , then:

$$\partial^2 f/\partial z^2 = (1 - u_s^2/(2i_0))^{6/25} \tau_s;$$

the left side of Eq. (17) is:

$$-\frac{1}{2} f \frac{\partial^2 f}{\partial z^2} = -\frac{1}{2} \left( \int_0^z u_s(l, z') dz' \right) \left( 1 - \frac{u_s^2}{2i_0} \right)^{6/25} \tau_s;$$

and the right side of Eq. (17) becomes:

$$\frac{\partial \tau_s}{\partial z} = \frac{\partial \tau_s}{\partial u_s} \frac{\partial u_s}{\partial z} = \left( 1 - \frac{u_s^2}{2i_0} \right)^{6/25} \tau_s \frac{\partial \tau_s}{\partial u_s}.$$



Thus, Eq. (17), in terms of  $\tau_s$  and  $u_s$ , allows the elimination of the factor  $(\sigma^{6/25} \tau_s)$ , present on both sides:

$$-\frac{1}{2} \int_0^z u_s(l, z') dz' (\sigma^{6/25} \tau_s) = c_1^{-1} c_2 c_3 \sigma_0^{\frac{b}{(b-1)}-1} (\sigma^{6/25} \tau_s) \frac{\partial \tau_s}{\partial u_s}. \tag{20}$$

A derivation with respect to  $z$  on both sides of Eq. (20) leads to Eq. (16).  $\square$

#### 4. Dorodnitzyn's Shear Stress Limit Formula

**Theorem 5.** *Under the same hypotheses of Theorem 4, let  $R \xrightarrow{\phi^\epsilon} R^\epsilon$  for  $\epsilon = h/L > 0$ , where we have  $(x, y) \xrightarrow{\phi^\epsilon} (x/L, y/(L\epsilon))$ ,  $(x/L, y/(L\epsilon)) = (x^*, y^*)$ . Furthermore, assume  $\partial u/\partial y(x, y) > 0$  for each Lebesgue point  $(x, y) \in R$ . Then, there is a limit  $u^*$ ,  $u^* \in W^{1,2}(R)$ ,  $u^* = \lim_{\epsilon \rightarrow 0} u^\epsilon$  of  $u^\epsilon = (1/L) u$ , such that:*

$$\frac{\partial}{\partial u_s} \left( 1 - \frac{(u^*)^2}{2i_0} \right)^{19/25} = 0. \tag{21}$$

*Proof.* Let  $\sigma^\epsilon = 1 - ([Lu^\epsilon]^2 / 2i_0)$ ,

$$\begin{aligned} \tau_s &= (1 - u_s^2 / (2i_0))^{(19/25)-1} \partial u_s / \partial z \\ &= \tilde{c} x^{1/2} \tau, \end{aligned}$$

where  $\partial u_s / \partial z = \ell^{1/2} \rho^{-1} \partial u / \partial y$ ,  $\tilde{c} = c_1^{1/2} c_2^{-1} \sigma_0^{1/2-b/(b-1)}$ , and  $\tau = \mu \partial u / \partial y$ . Thus, Eq. (16), in terms of  $\epsilon$ , becomes:

$$\begin{aligned} \epsilon \tilde{c} x^{1/2} (\sigma^\epsilon)^{19/25} \frac{\partial \tau_s}{\partial y} \left( \frac{\partial u^\epsilon}{\partial y^*} \right)^{-1} - \epsilon^2 \left( \frac{\partial \tau_s}{\partial y} \right)^2 \left( \frac{\partial u^\epsilon}{\partial y^*} \right)^{-2} \\ = -\frac{1}{2} c_1 c_2^{-1} c_3^{-1} \sigma_0^{1-\frac{b}{(b-1)}} u_s \left( 1 - \frac{u_s^2}{2i_0} \right)^{-6/25}. \end{aligned} \tag{22}$$

In a previous article [12], we showed that, under these circumstances,  $\|\nabla u^\epsilon\|_{L^2(R)} \leq (c_2 U^3)/(2C)$  for a constant  $C$  that is independent of the parameter  $\epsilon$ . This way, the sequence  $(u^\epsilon)$  is bounded in the Sobolev Space  $W^{1,2}(R)$ . Then, the Rellich-Kondrachov compactness theorem [19, p.173,178] implies that there is a subsequence that converges strongly in  $L^2(R)$ , and the sequence  $\partial u^\epsilon / \partial y^*$

converges weakly in  $L^2(R)$  to a generalized derivative  $\partial u^*/\partial y^*$  of the limit  $u^* \in L^2(R)$ . Hence,  $u^*$  is a weak solution of Eq. (22), in  $L^2(R)$  when the parameter  $\epsilon$  tends to 0.  $\square$

## 5. Conclusion

It is possible to deduce approximate shear stress formulas from the Dorodnitsyn's gaseous boundary layer model and a Reynolds' Limit Formula developed through a small parameter statement of the problem without taking away the convective derivative non-linear term of the conservation of momentum equation. These estimates provide a new family of deterministic boundary layer separation models to be analysed.

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